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## Weak-field gravity of circular cosmic strings

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A weak-field solution of Einstein's equations is constructed. It is generated by a circular cosmic string externally supported against collapse. The solution exhibits a conical singularity, and the corresponding deficit angle is the same as for a straight string for the same linear energy density. This confirms the deficit-angle assumption made in the Frolov-Israel-Unruh derivation of the metric describing a string loop at a moment of time symmetry.

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### I. INTRODUCTION

In a gauge theory with gauge group  $G$  spontaneously broken via the Higgs mechanism to a subgroup  $H$ , the Higgs field must lie on the manifold of degenerate vacua  $G/H$  on the circle at spatial infinity. Thus if  $\pi_1(G/H)$  is nontrivial, stringlike configurations with finite energy per unit length will exist for which the Higgs field at spatial infinity winds nontrivially around  $G/H$ . Such so-called cosmic strings have received much attention in recent years, especially as providing possible mechanisms for the formation of large-scale structure in the Universe [1]. Accretion wakes have been proposed to form behind infinitely long relativistic strings, which may help explain the observed large-scale filaments. Moreover, Vilenkin [2] has suggested that long-lived closed loops of string might act as seeds for galaxy and cluster formation.

The derivation of the gravitational field outside a cosmic string, in the weak-field approximation, was carried out by Vilenkin [3], who investigated the case of an infinitely thin straight string. The main features of this solution are that the spacetime exterior to the string is flat, and that all along the string, the spacetime has a conical singularity with deficit angle  $\delta\phi = 8\pi G\mu$ , where  $\mu$  is the linear energy density. (The solution to the exact Einstein equations has since been derived [4].)

Closed string loops have been studied by Frolov, Israel, and Unruh (FIU) [5]. Using the initial-value formulation [6], they constructed a family of momentarily stationary circular loops, which are considered as thin loops of string either at the time of formation or at the turning point between expansion and collapse. One of the

characteristics of the FIU method, which has the merit of taking into account the exact nonlinear field equations, is that it imposes *a priori* that all points on the circular string be conical singularities of spacetime with deficit angle  $\delta\phi = 8\pi G\mu$ . This is a reasonable assumption because of the observation [5] that an infinitely thin circular string, when viewed from arbitrarily close to the core, is indistinguishable from an infinitely thin straight string, and that all the points on the circular string may thus be assumed to be conical singularities with the same angular deficit as for a straight string of equal linear energy density.

The question we ask is the following: instead of making the above assumption, can one deduce from the field equations, at least in the weak-field limit, the fact that all the points on the circular string are conical singularities? As a first step in this direction one may construct, without any special hypothesis on conical singularities, a solution produced by a circular source generated by a stress-energy tensor obtained by adapting the method used by Vilenkin [3] for a straight string. This solution could then be examined for possible conical singularities, and the corresponding angular deficits, if any, could be calculated. We carry out such an investigation in this paper.

An important difference between our analysis and that of FIU is that we seek a solution which is stationary, whereas FIU consider a circular string loop which is momentarily at rest. Conceivably, a nonstationary solution at a moment of time symmetry may differ from a stationary solution by the presence of gravitational waves, but FIU considered precisely a class of particular solutions devoid of such free gravitational radiation. Therefore, a comparison between the two cases is physically reasonable. Of course an isolated stationary circular string is unphysical, since it would tend to collapse. To overcome this problem, radial stresses, the values of which are determined by stress-energy conservation conditions, are introduced to support the string. It will be seen that these stresses do not contribute to the value of the angular deficit.

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In Sec. II, we establish the appropriate form of the stress-energy tensor for this problem. We derive and solve the weak-field Einstein equations in Secs. III and IV. In Sec. V, we demonstrate that the solution does indeed exhibit conical singularities with the same angular deficit as for a straight string with the same linear energy density, thus fully supporting and agreeing with the FIU hypothesis.

In this paper, we use units in which  $\hbar=c=1$ , take the metric to have signature  $(-, +, +, +)$  and adopt the geometrical conventions of [7].

## II. THE STRESS-ENERGY TENSOR AND THE METRIC

In this section, we establish the stress-energy tensor  $T_\mu^\nu$  for a loop of cosmic string arising from a spontaneously broken gauge theory. In the case of an infinitely long straight string, all components of the stress-energy tensor are localized on the core of the string, that is the region of spacetime where the Higgs field unwinds and thus does not lie on the manifold of degenerate vacua. The radius of this region is of the order of the Compton wavelength of the Higgs field [8], generically a microscopic distance. Thus, when considering the gravitational effects of the string, it is physically reasonable to make the thin-string approximation where the stress-energy tensor is localized on an infinitely thin line.

A circular source produces an axially-symmetric gravitational field, and the most general stationary metric of this type may be cast in the form [7]

$$ds^2 = -e^{2\nu} dt^2 + e^{2\xi - 2\nu} r^2 d\phi^2 + e^{2\eta - 2\nu} (dr^2 + dz^2), \quad (2.1)$$

where the three functions  $\nu$ ,  $\xi$ , and  $\eta$  depend only on  $r$  and  $z$ , and  $x^\alpha \equiv (t, \phi, r, z)$  denotes cylindrical coordinates.

The form of the Einstein tensor for this metric, together with the Einstein field equations  $G_\mu^\nu = -\kappa T_\mu^\nu$  ( $\kappa = 8\pi G$ ) implies that the most general stress-energy tensor compatible with (2.1) is

$$T_\mu^\nu = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \Delta & \epsilon \\ 0 & 0 & \epsilon & \beta \end{pmatrix}, \quad (2.2)$$

where  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\Delta$ , and  $\epsilon$  are solely functions of  $r$  and  $z$ .

In passing to the limit of an infinitely thin loop of string with radius  $a$ , lying in the  $x$ - $y$  plane and centered at the origin, we obtain (with the exception of  $T_r^r$ , dealt with below) the stress-energy tensor as

$$T_\mu^\nu = \bar{T}_\mu^\nu \delta(r-a) \delta(z), \quad (2.3)$$

where  $\bar{T}_\mu^\nu$  is the cross-sectional integral of  $T_\mu^\nu$ . Therefore this limit yields, in obvious notation,

$$\alpha = \bar{\alpha} \delta(r-a) \delta(z), \quad (2.4)$$

$$\beta = \bar{\beta} \delta(r-a) \delta(z), \quad (2.5)$$

$$\gamma = \bar{\gamma} \delta(r-a) \delta(z), \quad (2.6)$$

$$\epsilon = \bar{\epsilon} \delta(r-a) \delta(z). \quad (2.7)$$

However, such a localization does not occur for the radial stress  $T_r^r \equiv \Delta$  required to support the ring. Given that the ring is *externally* supported, this radial stress is not confined to the core. (In what follows, we choose to support the ring from infinity.)

Following Vilenkin, we can show that  $\bar{\beta} = \bar{\epsilon} = 0$ . Indeed, taking an arbitrary string cross-section  $\mathcal{M} = \{(\phi, r, z) : \phi = \text{const}\}$  and using stress-energy conservation ( $T_\mu^\nu|_{;\nu} = 0$ ), we have

$$0 = \int_{\mathcal{M}} T_\mu^\nu|_{;\nu} x^\lambda dr dz. \quad (2.8)$$

In the weak-field limit, the components of the stress-energy tensor (2.2) are taken to be of the same order of magnitude as the metric functions  $\nu$ ,  $\xi$ , and  $\eta$ , which are assumed to be small compared to unity. Thus, to lowest order, the only Christoffel symbols for the metric (2.1) that contribute to the covariant derivative of the stress-energy tensor are  $\Gamma_{\phi\phi}^r = -r$  and  $\Gamma_{r\phi}^\phi = \Gamma_{\phi r}^\phi = 1/r$ . Taking  $\mu = z$  and noting that  $T_z^t = T_z^z = 0$ , we may integrate by parts to get

$$0 = \bar{T}_z^r \delta_r^\lambda + \bar{T}_z^z \delta_z^\lambda \quad (2.9)$$

(no sum over  $r$  and  $z$ ). There is no boundary contribution since  $T_z^\lambda$  is localized on the string core. We emphasize that, in contrast with the case of the straight string, a similar argument would fail to show that  $\bar{T}_r^r = 0$  because  $T_r^r \equiv \Delta$  is nonzero outside the core. We obtain the desired result ( $\bar{\epsilon} = \bar{\beta} = 0$ ) by taking  $\lambda = r, z$  in (2.9). The last undetermined function,  $\Delta$ , in the stress-energy tensor can be found in terms of  $\gamma$  from the stress-energy conservation equations, and will be calculated in the next section.

At this stage, we have deduced that the stress-energy tensor for a thin circular matter source supported by external radial stresses is given by (2.2) with  $\beta = \epsilon = 0$ , and  $\alpha$ ,  $\Delta$  and  $\gamma$  as above. We rewrite  $\bar{\alpha} = -\mu$  and  $\bar{\gamma} = k$ , where  $\mu$  and  $k$  are the linear energy density and the azimuthal stress per unit length, respectively. Henceforth, we make the thin-string approximation and use this form of the stress-energy tensor.

To specialize to the case of a cosmic string in a spontaneously broken gauge theory, we would apply the equation of state [3] for string matter  $k = -\mu$ . (This equation of state is dictated by Lorentz invariance in the straight-string case; for a circular string, the azimuthal stress  $T_\phi^\phi$  plays the same role as that of the longitudinal stress  $T_z^z$  for a straight string.) We will, however, keep  $\mu$  and  $k$  as two independent parameters. This has the advantage that we will then be able to compare and contrast ordinary matter, given by  $k$  positive and small compared to  $\mu$ , with string matter, given by  $k = -\mu$ .

## III. THE FIELD EQUATIONS AND STRESS-ENERGY CONSERVATION

In the weak-field approximation, the nonzero components of the Einstein tensor [9] for the metric (2.1) are, upon retaining only first-order terms in  $\nu$ ,  $\xi$ , and  $\eta$ ,

$$G_t^t = 2\nabla^2 \nu - \nabla^2 \xi - \frac{1}{r} \xi_r - \bar{\nabla}^2 \eta, \quad (3.1)$$

$$G_\phi^\phi = -\tilde{\nabla}^2 \eta, \quad (3.2)$$

$$G_r^r = -\xi_{zz} - \frac{1}{r} \eta_r, \quad (3.3)$$

$$G_z^z = -\xi_{rr} + \frac{1}{r} (\eta_r - 2\xi_r), \quad (3.4)$$

$$G_r^z = \xi_{zr} + \frac{1}{r} (\xi_z - \eta_z), \quad (3.5)$$

where  $\tilde{\nabla}^2 \equiv \partial_r^2 + \partial_z^2$ , and  $\nabla^2 \equiv \partial_r^2 + (1/r)\partial_r + \partial_z^2$  is the flat-space Laplacian operator. (We stress that by “weak-field approximation,” we mean retaining only first-order terms in  $v$ ,  $\eta$  and  $\xi$ .) After a short calculation, the field equations reduce to

$$\nabla^2 v = 4\pi G(-\alpha + \gamma + \Delta), \quad (3.6)$$

$$\tilde{\nabla}^2 \eta = 8\pi G \gamma, \quad (3.7)$$

$$\nabla^2 \xi + \frac{1}{r} \xi_r = 8\pi G \Delta, \quad (3.8)$$

$$\eta_r = r \xi_{rr} + 2\xi_r, \quad (3.9)$$

$$\eta_z = r \xi_{zr} + \xi_z. \quad (3.10)$$

We integrate (3.9) and (3.10) to get

$$\eta = \frac{\partial}{\partial r} (r \xi) + \eta_0, \quad (3.11)$$

where  $\eta_0$  is an arbitrary constant to be determined later.

The conservation of energy and momentum [or equivalently the compatibility conditions for (3.7)–(3.10)] places constraints on the allowable components of  $T_\mu^\nu$ , which in the weak-field limit read

$$\gamma = \frac{\partial}{\partial r} (r \Delta). \quad (3.12)$$

The solution of (3.12) for  $\Delta$ , with the azimuthal stress  $\gamma$  given by (2.6) and corresponding to supporting the string from infinity, is

$$\Delta(r, z) = \frac{k}{r} \Theta(r - a) \delta(z), \quad (3.13)$$

in which  $\Theta$  denotes the Heaviside step function. Furthermore, Eqs. (3.11) and (3.12) imply that (3.7) and (3.8) are equivalent. We may therefore discard (3.8), retain (3.7) to find  $\eta$ , and use (3.11) to express  $\xi$  in terms of  $\eta$ .

Combining all the results, the field equations are equivalent to solving the following set of equations for  $v$ ,  $\eta$ , and  $\xi$ :

$$\nabla^2 v = 4\pi G \left[ (\mu + k) \delta(r - a) + \frac{k}{r} \Theta(r - a) \right] \delta(z), \quad (3.14)$$

$$\tilde{\nabla}^2 \eta = 8\pi G k \delta(r - a) \delta(z), \quad (3.15)$$

$$\frac{\partial}{\partial r} (r \xi) = \eta - \eta_0. \quad (3.16)$$

These equations involve the dimensionless quantities  $G\mu$  and  $Gk$ . For a cosmic string,  $k = -\mu$ , whereas for ordinary matter,  $k \ll \mu$ . The largest value of  $\mu$  in a physically relevant theory occurs for grand unified theory (GUT)

strings, for which  $G\mu \approx 10^{-6}$  [3], validating the weak-field approximation as an expansion of  $v$ ,  $\xi$ , and  $\eta$  in powers of  $G\mu$ .

#### IV. SOLUTION OF THE FIELD EQUATIONS

We note that (3.14) is Poisson's equation for the Newtonian potential  $v$ . The solution is most easily found [10] in toroidal coordinates  $(\phi, \sigma, \psi)$ . The latter are related to cylindrical coordinates  $(\phi, r, z)$  by

$$z = aN^{-2} \sin \psi, \quad (4.1)$$

$$r = aN^{-2} \sinh \sigma \quad (0 \leq \sigma \leq \infty, \quad |\psi| \leq \pi),$$

where  $N^2 \equiv N^2(\sigma, \psi) \equiv \cosh \sigma - \cos \psi$ . The surfaces  $\sigma = \sigma_0 \equiv \text{const}$  are tori whose generating circles have radii  $a \text{csch} \sigma_0$  and  $a \coth \sigma_0$ . In particular, the ring  $r = a, z = 0$  is now given by  $\sigma = \infty$ . (See Fig. 1 in Ref. [5].) In toroidal coordinates, the metric (2.1) reads

$$ds^2 = -e^{2v} dt^2 + a^2 N^{-4} \sinh^2 \sigma e^{2\xi - 2v} d\phi^2 + a^2 N^{-4} e^{2\eta - 2v} (d\sigma^2 + d\psi^2), \quad (4.2)$$

where  $v$ ,  $\xi$ , and  $\eta$  depend on  $\sigma$  and  $\psi$ .

The potential  $v$  can be split into two parts,  $v = v_1 + v_2$ , where  $v_1$  and  $v_2$  satisfy separate Poisson equations

$$\nabla^2 v_1 = 4\pi G(\mu + k) \delta(r - a) \delta(z), \quad (4.3)$$

$$\nabla^2 v_2 = 4\pi G \frac{k}{r} \Theta(r - a) \delta(z). \quad (4.4)$$

The function  $v_1$  is the gravitational potential produced by a circular ring of matter. This problem is known [10], and the corresponding solution is

$$v_1 = -2^{3/2} G(\mu + k) N(\sigma, \psi) K(\tanh(\frac{1}{2}\sigma)) / \cosh(\frac{1}{2}\sigma), \quad (4.5)$$

where  $K$  denotes the complete elliptic integral of the first kind.

The formal solution of Poisson's equation with source  $\rho$  is given by

$$v_2(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}_0) dV_0}{|\mathbf{r} - \mathbf{r}_0|}. \quad (4.6)$$

Moreover, the Green's function in toroidal coordinates has the expansion [11]

$$\begin{aligned} \frac{\pi a}{|\mathbf{r} - \mathbf{r}_0|} &= N(\sigma, \psi) N(\sigma_0, \psi_0) \\ &\times \sum_{m,n=0}^{\infty} C_{mn} \cos m(\phi - \phi_0) \cos n(\psi - \psi_0) \\ &\times P_{n-1/2}^m(\cosh \sigma_<) Q_{n-1/2}^m(\cosh \sigma_>), \end{aligned} \quad (4.7)$$

where  $\sigma_> \equiv \max\{\sigma, \sigma_0\}$ ,  $\sigma_< \equiv \min\{\sigma, \sigma_0\}$ , and the functions  $P_{n-1/2}^m$  and  $Q_{n-1/2}^m$  are toroidal Legendre functions [12]. The numerical coefficients are

$$C_{mn} = (-1)^m \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})} \epsilon_n \epsilon_m, \quad (4.8)$$

where  $\epsilon_n \equiv 2 - \delta_n^0$ .

In toroidal coordinates, we have

$$\Theta(r-a)\delta(z) = a^{-1} N^2(\sigma, \psi) \delta(\psi). \quad (4.9)$$

Inserting (4.9), together with (4.7) and (4.8), into (4.6), we find the following expression for  $\nu_2$  in toroidal coordinates:

$$\begin{aligned} \nu_2 = & -2GkN(\sigma, \psi) \\ & \times \sum_{n=0}^{\infty} [F_n(\sigma) P_{n-1/2}(\cosh \sigma) \\ & + G_n(\sigma) Q_{n-1/2}(\cosh \sigma)] \epsilon_n \cos n\psi, \end{aligned} \quad (4.10)$$

$$F_n(\sigma) \equiv \int_0^\sigma d\tau N^{-1}(\tau, 0) Q_{n-1/2}(\cosh \tau), \quad (4.11)$$

$$G_n(\sigma) \equiv \int_0^\sigma d\tau N^{-1}(\tau, 0) P_{n-1/2}(\cosh \tau), \quad (4.12)$$

and we use the abbreviations  $P_{n-1/2} \equiv P_{n-1/2}^0$ ,  $Q_{n-1/2} \equiv Q_{n-1/2}^0$  for the toroidal harmonic functions.

We now turn to the solution of the  $\eta$  equation. Writing (3.15) in full

$$\frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial z^2} = 8\pi Gk \delta(r-a) \delta(z), \quad (4.13)$$

we can interpret it (after formally replacing  $r$  by  $x$ ) as the potential produced by a uniform thin rod lying in the  $x$ - $y$  plane, parallel to the  $y$  axis at  $x=a$ . This is a standard problem in electrostatics [13], and in cylindrical coordinates the solution reads

$$\eta(r, z) = 2Gk \ln \{ [(r-a)^2 + z^2] / R_0^2 \}, \quad (4.14)$$

where  $R_0$  is a constant to be discussed later.

The last remaining metric function to be found is  $\zeta$ . Solving (3.16) with the above  $\eta$ , we obtain in cylindrical coordinates

$$\zeta(r, z) = -\eta_0 + \frac{2Gk}{r} \left\{ r \ln \left[ \frac{(r-a)^2 + z^2}{R_0^2} \right] - a \ln \left[ \frac{(r-a)^2 + z^2}{a^2 + z^2} \right] - 2r + 2z \left[ \arctan \left[ \frac{r-a}{z} \right] + \arctan \left[ \frac{a}{z} \right] \right] \right\}, \quad (4.15)$$

in which an arbitrary integration function of  $z$  has been determined by the requirement that the solution be regular on the  $z$  axis. Calculating the angular deficit about the  $z$  axis yields

$$\delta\phi = 2\pi(1 - e^{-\eta_0}). \quad (4.16)$$

Consequently, demanding that there be no conical singularity along the  $z$  axis imposes  $\eta_0 = 0$ .

Having determined  $\nu$ ,  $\eta$ , and  $\zeta$ , we have formal solutions of the weak-field Einstein equations. We must now check whether these solutions are valid within the weak-field approximation. It is clear from (3.14) and (3.15) that  $\nu$  and  $\eta$  are determined only up to an additive constant. Moreover, by virtue of (3.16), any constant added to  $\zeta$  may be absorbed in a redefinition of  $\eta_0$ , leaving the structure of the equation unchanged. This freedom of an additive constant in each of  $\nu$ ,  $\eta$ , and  $\zeta$  enables one to ensure that the weak-field approximation is valid at least near the string, the region of spacetime that will be relevant when investigating conical singularities and the corresponding deficit angle in the next section. (In the standard weak-field approximation in Cartesian coordinates around Minkowski space, these constants are naturally chosen so that the potentials vanish at infinity.) Furthermore, all future results will be seen to be independent of the particular choice of these additive constants.

Within the model of an infinitely thin string, there is no natural way to determine the additive constants, since any choice leads to some of the potentials becoming infinite near the string core, invalidating the weak-field

approximation. [For instance, irrespective of the value of  $R_0$  in (4.14),  $\eta$  is infinite at the ring.] However, a physical string always has a nonvanishing thickness, something, strictly speaking, beyond the realm of our thin-string model. To give, nevertheless, a value to these constants, we may formally consider a slightly fattened string of core radius  $R_{\text{core}}$ , and fix the additive constants by demanding that the potentials be small at distances of order  $R_{\text{core}}$  from the infinitely thin ring. (For instance, taking  $R_0 = R_{\text{core}}$  in (4.14) leads to  $\eta \approx 0$  at distances of the order of  $R_{\text{core}}$  from the infinitely thin string.) In what follows, we stay within the infinitely thin string model, and we implement the weak-field approximation by the requirement that  $\nu, \eta, \zeta \ll 1$  on distance scales of order  $R_{\text{core}}$  from the infinitely thin string. The actual value selected for  $R_{\text{core}}$  does not follow from the thin-string model, and must be supplied by the underlying physical theory. We emphasize that we are mathematically dealing with an infinitely thin string core, with only the faint memory of its physical origin in the choice of  $R_{\text{core}}$ . It is, however, important to stress that all future results will be independent of the particular choice of  $R_{\text{core}}$ .

## V. ANGULAR DEFICIT

To check the metric (4.2) for conical singularities on the ring  $\sigma = \infty$ , we must examine circles  $\sigma = \sigma_0$  at constant  $t$  and  $\phi$  around the ring, and calculate the ratio of the proper perimeter to the proper radius in the limit that the proper radius tends to zero. For  $t$  and  $\phi$  constant, the metric (4.2) becomes

$$ds^2 = a^2 N^{-4} e^{2\eta-2\nu} (d\sigma^2 + d\psi^2), \quad (5.1)$$

and the angular deficit  $\delta\psi$  is given by

$$\delta\psi = 2\pi - \lim_{\sigma_0 \rightarrow \infty} \frac{\int_{-\pi}^{\pi} d\psi (N^{-2} e^{\eta-\nu})|_{\sigma=\sigma_0}}{\int_{\sigma_0}^{\infty} d\sigma N^{-2} e^{\eta-\nu}}. \quad (5.2)$$

From this formula, one sees that the value of  $\delta\psi$  is independent of any additive constants that might appear in  $\nu$  or  $\eta$ , as emphasized above. In this calculation we are thus free to ignore such additive constants, and this will be done below.

In order to evaluate the limit appearing in (5.2), we require the asymptotic form of  $\nu$  and  $\eta$  for large values of  $\sigma$ . The contribution of  $\nu_1$  from (4.5) to the asymptotic behavior of  $\nu$  for  $\sigma$  tending to infinity is readily found to be

$$\nu_1 \rightarrow -2G(\mu + k)\sigma, \quad \sigma \rightarrow \infty. \quad (5.3)$$

(In establishing this formula, we use the asymptotic expression  $K(\tanh(\sigma/2)) \rightarrow \sigma/2$  when  $\sigma$  tends to infinity [14].) In the Appendix, we show that  $\nu_2 \rightarrow 0$  as  $\sigma \rightarrow \infty$ . Thus, the asymptotic value of  $\nu$  does not depend on  $\nu_2$ , and this shows that the radial stresses  $T_r^r = (k/r)\Theta(r-a)\delta(z)$  do not contribute to the potential near the ring. Furthermore, the asymptotic behavior of  $\eta$  near the ring is

$$\eta(\sigma, \psi) \rightarrow -4Gk\sigma, \quad \sigma \rightarrow \infty. \quad (5.4)$$

With (5.3) and (5.4), it is straightforward to evaluate the integral (5.2) for the deficit angle  $\delta\psi$  as

$$\delta\psi = 4\pi G(\mu - k), \quad (5.5)$$

which will be analyzed in the conclusion.

The same result for the angular deficit can be obtained in a different manner, which also sheds light on the spacetime structure near the ring in the weak-field limit. Using the asymptotic forms of  $\nu$  and  $\eta$  near the ring, the metric (5.1) becomes

$$ds^2 = 4a^2 e^{-2\sigma} e^{4G(\mu-k)\sigma + 2b} (d\sigma^2 + d\psi^2), \quad \sigma \rightarrow \infty, \quad (5.6)$$

where  $b$  denotes the dimensionless combination of all additive constants appearing in the potentials. If we now seek coordinates  $(\sigma', \psi')$ , so that, near the ring, the metric is locally flat,

$$ds^2 = 4a^2 e^{-2\sigma'} (d\sigma'^2 + d\psi'^2), \quad \sigma \rightarrow \infty, \quad (5.7)$$

we see that we must take

$$e^b e^{-\{(1-\lambda)\sigma\}} d\sigma = e^{-\sigma'} d\sigma', \quad (5.8)$$

where  $\lambda \equiv 2G(\mu - k)$ . Thus, to first order in  $\lambda$ , this yields

$$\sigma' = (1 - \lambda)\sigma - \lambda - b. \quad (5.9)$$

Consequently, if we define  $\psi' = (1 - \lambda)\psi$ , we note that we do obtain the metric in the form (5.7). It is then clear that  $\psi'$  runs from 0 to  $2\pi(1 - \lambda)$ , and that we thus recover the deficit angle

$$\delta\psi = 4\pi G(\mu - k) \quad (5.10)$$

already found in (5.5), once again seen to be independent of the choice of the additive constants. A similar method was employed by Vilenkin [3].

## VI. CONCLUSION

In this paper, we constructed a stationary, axially-symmetric metric satisfying Einstein's weak-field equations for a source describing an infinitely thin ring of radius  $a$ , linear mass density  $\mu$ , and azimuthal stress per unit length  $k$ . The conservation of stress-energy required the presence of a radial stress

$$T_r^r = \frac{k}{r} \Theta(r - a) \delta(z) \quad (6.1)$$

to support the ring. The metric obtained exhibits a conical singularity along the ring, with deficit angle

$$\delta\psi = 4\pi G(\mu - k). \quad (6.2)$$

In the case of a ring of pressureless dust ( $k = 0$ ), a conical singularity arises, with deficit angle  $4\pi G\mu$ . This is usually interpreted by saying that, because of the presence of the singularity, such a source cannot remain stationary and must collapse.

When the ring is made of "ordinary" matter ( $k > 0$ ,  $k \ll \mu$ ), the conical singularity is still present but is less severe than for pressureless dust. Such a source is also considered as nonphysical since, unless the (positive) pressure equals the energy density, the angular deficit  $\delta\psi$  is still nonzero.

When the ring is made of "string" matter ( $k = -\mu$ ), we recover Vilenkin's result  $\delta\psi = 8\pi G\mu$ . This result is interesting for two reasons: first, it shows that one-half of the angular deficit of the string source is of "nonstring" origin (since one-half of the effect remains for pressureless dust); second, it establishes *from the field equations*, within the thin-string model, that indeed, a conical singularity is present along a circular string, and that the angular deficit takes the same value as for a straight string of the same linear mass density. This was an *assumption* made by FIU in their investigation of circular cosmic strings at an instant of time symmetry. Our formalism provides a weak-field proof of the validity of the FIU hypothesis.

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## APPENDIX: ASYMPTOTIC BEHAVIOR OF $\nu_2$ NEAR THE RING

Establishing the asymptotic behavior of the potential  $\nu_2$  (4.10) near the ring, namely, for  $\sigma$  tending to infinity,

is a delicate matter. This is due to the fact that the source, being of the form  $(k/r)\Theta(r-a)\delta(z)$ , decreases very slowly at infinity when the radius  $r$  increases without limit. The difficulty at infinity in  $r$  translates into a difficulty at 0 in  $\sigma$ .

For the calculation of  $F_n(\sigma)$  of (4.11), however, the point  $\sigma=0$  does not belong to the domain of integration and one finds, after replacing  $Q_{n-1/2}(\cosh\sigma)$  by its asymptotic value [15] and performing the integration,

$$F_n(\sigma) \rightarrow (2\pi)^{1/2} \Gamma(n + \frac{1}{2}) e^{-(n+1)\sigma} / (n+1)! . \quad (\text{A1})$$

On the other hand, a problem does arise in the calculation of  $G_n(\sigma)$ : Near  $\tau=0$ , the toroidal harmonics  $P_{n-1/2}(\cosh\tau)$  tend to 1 for all  $n$  [16], whereas the factor  $N^{-1}(\tau, 0)$  appearing in the integrand of (4.12) tends to  $2^{1/2}\tau^{-1}$ . As a result,  $G_n$  diverges, and a cutoff  $\varepsilon$  must be introduced. If this is done, the integrand is finite everywhere, and the regularized  $G_n(\sigma)$ , denoted by  $G_n^R(\sigma)$ , reads

$$G_n^R(\sigma) \equiv \int_{\varepsilon}^{\sigma} d\tau N^{-1}(\tau, 0) P_{n-1/2}(\cosh\tau) . \quad (\text{A2})$$

To go further, one must consider the asymptotic value of  $P_{n-1/2}(\cosh\sigma)$ .

For  $n=0$ , the corresponding toroidal harmonic is expressible in terms of the complete elliptic integral of the first kind [17] as

$$P_{-1/2}(\cosh\sigma) = 2\pi^{-1} K(\tanh(\frac{1}{2}\sigma)) / \cosh(\frac{1}{2}\sigma) . \quad (\text{A3})$$

For very large  $\sigma$ , this is asymptotically given [14] by

$$P_{-1/2}(\cosh\sigma) \rightarrow 2\pi^{-1} \sigma e^{-\sigma/2} . \quad (\text{A4})$$

For  $n \geq 1$ , the asymptotic expression [18] of  $P_{n-1/2}(\cosh\sigma)$  is

$$P_{n-1/2} \rightarrow \pi^{-1/2} (n-1)! e^{(n-1/2)\sigma} / \Gamma(n + \frac{1}{2}), \quad n \geq 1 . \quad (\text{A5})$$

When the asymptotic expressions (A4) and (A5) are inserted into the integral, (A2) for  $G_n^R(\sigma)$ , the following results are obtained:

$$G_0^R(\sigma) \rightarrow A(\varepsilon), \quad A(\varepsilon) \equiv \int_{\varepsilon}^{\infty} d\tau N^{-1}(\tau, 0) P_{-1/2}(\cosh\tau) , \quad (\text{A6})$$

$$G_1^R(\sigma) \rightarrow (2/\pi)^{1/2} \sigma / \Gamma(\frac{3}{2}) \quad (\text{A7})$$

$$G_n^R(\sigma) \rightarrow (2/\pi)^{1/2} (n-2)! e^{(n-1)\sigma} / \Gamma(n + \frac{1}{2}), \quad n \geq 2 , \quad (\text{A8})$$

so that finally, the asymptotic form of the series (4.10) becomes

$$\nu_2 \rightarrow -Gk [2^{1/2} \pi A(\varepsilon) + o(e^{-\sigma})] . \quad (\text{A9})$$

As one can see, for every nonvanishing value of  $\varepsilon$ , the potential  $\nu_2$  tends to a constant on the ring, that is, for  $\sigma$  tending to infinity. The fact that  $\nu_2$  is asymptotically constant is sufficient to establish the result (5.5) of the deficit angle, irrespective of the value of the constant. It is, however, possible to go one step further and renormalize the potential so that  $\nu_2$  vanishes at the ring. This amounts to subtracting from  $\nu_2$  the  $(\sigma, \psi)$ -independent term  $2^{1/2} \pi A(\varepsilon)$ . [Although, when the cutoff  $\varepsilon$  tends to 0, the constant  $A(\varepsilon)$  diverges, this procedure is nothing more than a conventional renormalization.]

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